Non associative renormalization group

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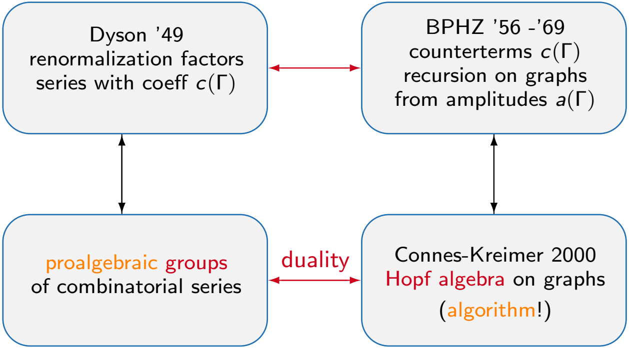
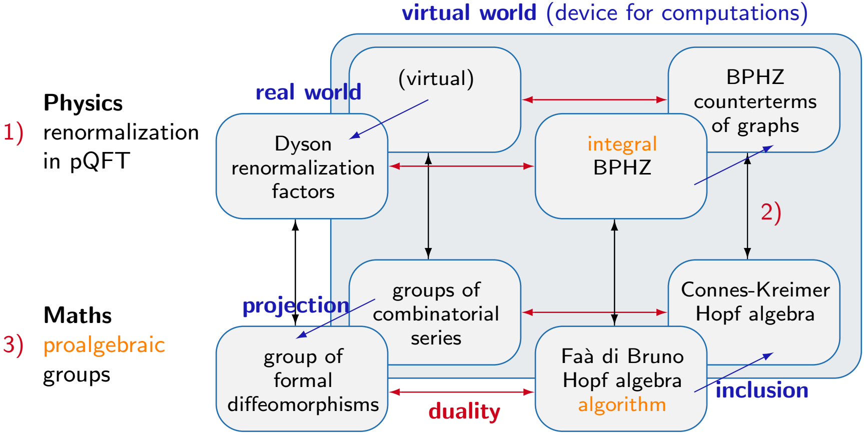
Motivation

Physics
renormalization
in pQFT

Maths
algebraic groups



Motivation and plan

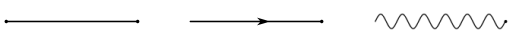


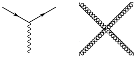
Pb: **duality** holds iff the coefficients, amplitudes and Hopf algebras are **commutative**, but **in QED and QCD amplitudes are matrices**.

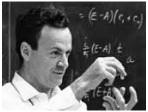
4) Extend **duality** to **non-commutative** algebras.

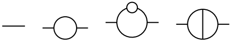
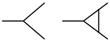
5) When **duality fails with groups**, extend to **loops** = **non-assocociative groups**.

- **Problems in QED** [1930's]: predictions on electron mass and charge need corrections
- **Feynman graphs** [1948]: $\mathcal{L}(\phi;\lambda) = \mathcal{L}_0(\phi) + \lambda\,\mathcal{L}_{int}(\phi)$

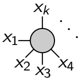
  $\mathcal{L}_0$ gives the free propagator  ...

  $\mathcal{L}_{int}$ gives vertices  ...

  $\Rightarrow$ Feynman graphs Γ, e.g. for ϕ^3:  or 

  with amplitude $a(\Gamma)$ = integral over internal points with Feynman rules.

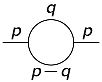
- **Green functions**: $G^{(k)}(x_1,\dots,x_k;\lambda) =$ 

$= \sum_{E(\Gamma)=k} a(\Gamma; x_1,\dots,x_k)\,\hbar^{L(\Gamma)}\lambda^{V(\Gamma)}$

- **Formal series in λ** with coefficients in $A = \mathbb{C}, M_4(\mathbb{C})\dots$ given by $\mathcal{L}_0$:

$$G_n^{(k)} = \sum_{\substack{V(\Gamma)=n\\ E(\Gamma)=k}} a(\Gamma)\,\hbar^{L(\Gamma)} \in A[\hbar] \quad\Longrightarrow\quad G^{(k)}(\lambda) = \sum_{n\geqslant 0} G_n^{(k)}\,\lambda^n \in A[\hbar][[\lambda]]$$

Renormalization

- **Divergent graphs**: 

$$= \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2+m^2} \frac{1}{(p-q)^2+m^2} \simeq \int_{|q|_{min}}^{\infty} d|q| \frac{1}{|q|} = \infty \ !$$

  Counterterms $c(\Gamma) = -$ divergent part (scalar in A)

  Amplitudes $a^{ren}(\Gamma) = a(\Gamma) + c(\Gamma) + \text{terms} \implies$ $\boxed{G^{ren}(\lambda) = \sum a^{ren}(\Gamma)\, \hbar^{L(\Gamma)} \lambda^{V(\Gamma)}}$

- **Dyson formula** [1949]:

  collect $c(\Gamma)$'s in few series $Z_i(\lambda)$ s.t. $\begin{array}{l} \phi_0 = \phi\, Z_3(\lambda)^{1/2} \\ \lambda_0 = \lambda\, Z_1(\lambda) Z_3(\lambda)^{-3/2} \end{array}$

  then $\boxed{\mathcal{L}^{ren}(\phi;\lambda) = \mathcal{L}(\phi_0;\lambda_0)}$ and $\boxed{G^{ren}(\lambda) = G\big(\lambda_0(\lambda)\big)\, Z_3(\lambda)^{-1/2}}$

  Renormalization factors: $Z(\lambda) = 1 + O(\lambda) \Rightarrow$ invertibile series with product

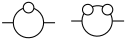
  Bare coupling: $\lambda_0(\lambda) = \lambda + O(\lambda^2) \Rightarrow$ formal diffeomorphism with substitution

- **Ren. group** (perturbative) = $\boxed{\text{bare coupling} \ltimes \text{ren. factors}}$ contains $(\lambda_0(\lambda), Z_i(\lambda))$

  Semidirect product $\boxed{(\lambda_0', Z') \bullet (\lambda_0(\lambda), Z(\lambda)) = \Big(\lambda_0'\big(\lambda_0(\lambda)\big)\,,\ Z'\big(\lambda_0(\lambda)\big)\, Z(\lambda)\Big)}$

  $\implies$ acts on $G(\lambda)$ by Dyson's formula $\boxed{G^{ren} = G \bullet (\lambda_0, Z)}$

2) Counterterms and Hopf algebras

- **BPHZ formula** ['57–'69]: recurrence on 1PI divergent subgraphs 

$$a^{ren}(\Gamma) = a(\Gamma) + c(\Gamma) + \sum_{(\gamma_i)} a(\Gamma_{/(\gamma_i)})\ c(\gamma_1)\cdots c(\gamma_r)$$

$$c(\Gamma) = -\text{Taylor}^{div(\Gamma)}\left[a(\Gamma) + \sum a(\Gamma_{/(\gamma_i)})\ c(\gamma_1)\cdots c(\gamma_r)\right]$$

$\gamma_1, \dots, \gamma_r \subset \Gamma$
1PI disjoint

- **Hopf algebra on Feynman graphs** [Connes-Kreimer '98-2000]:

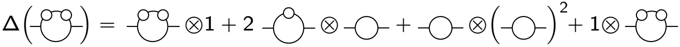
$$H_{\mathrm{CK}} = \mathbb{C}[\text{1PI } \Gamma] \quad \text{free commutative product}$$

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum \Gamma_{/(\gamma_k)} \otimes \gamma_1 \cdots \gamma_r$$

$$S(\Gamma) = -\left[\Gamma + \sum \Gamma_{/(\gamma_k)} S(\gamma_1)\cdots S(\gamma_r)\right]$$

Hopf algebra

multiplication	$m : H \otimes H \to H$	comultiplication	$\Delta : H \to H \otimes H$
unit	$u : \mathbb{K} \hookrightarrow H$	counit	$\varepsilon : H \to \mathbb{K}$
		antipode	$S : H \to H$

e.g. 

amplitudes = algebra maps $a, a^{ren} : H_{\mathrm{CK}} \to A[\hbar]$ related to coproduct Δ

counterterms = algebra map $c : H_{\mathrm{CK}} \to \mathbb{C} \subset A[\hbar]$ related to antipode S

3) Groups of series with coefficients in a commutative algebra A

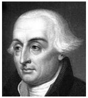
- **Proalgebraic group**: representable functor

$$G : \mathcal{C}om \to \mathcal{G}roups \qquad A \mapsto G(A) = \mathrm{Hom}_{\mathcal{C}om}(H, A)$$

H = coordinate ring of G gen. by coordinate functions $x_n(g) := g(x_n)$

- **Duality:** H is a Hopf algebra with $\Delta_H(x_n)(g, g') = x_n(gg')$
  G is the convolution group with $gg' = m_A(g \otimes g')\Delta_H$
- **Compact Lie groups** [Tannaka-Krein 1939]: H = representative fuctions
- **Formal diffeomorphisms** [Lagrange 1770, Faà di Bruno 1855]:

$$\mathrm{Diff}(A) = \Big\{ a(\lambda) = \sum a_n \lambda^{n+1} \,|\, a_0 = 1, a_n \in A \Big\} \qquad (a \circ b)(\lambda) = a\big(b(\lambda)\big)$$



- **Diffeographisms** [Connes-Kreimer 2000]:

$$\mathrm{Diff}_{\mathrm{CK}}(A) := \mathrm{Hom}_{\mathcal{C}om}(H_{\mathrm{CK}}, A) = \Big\{ a(\lambda) = \sum_{\Gamma} a_\Gamma \; \lambda^\Gamma \;\Big|\; a_\Gamma \in A \Big\}$$

$$(a \bullet b)(\lambda) = \sum_{\Gamma} \Big(a_\Gamma + b_\Gamma + \sum a_{\Gamma_{/(\gamma_k)}} \; b_{\gamma_1} \cdots b_{\gamma_r} \Big) \; \lambda^\Gamma$$

"virtual" series!

"λ^Γ" symbol

- **Virtual** $\twoheadrightarrow$ **Real:** projection

$$\mathrm{Diff}_{\mathrm{CK}}(A) \twoheadrightarrow \mathrm{Diff}(A), \;\; \lambda^\Gamma \mapsto \lambda^{V(\Gamma)}$$

- **In QFT:** need integral counterterms for

$$Z_k(\lambda) = 1 + \sum_{E(\Gamma)=k} \frac{c_k(\Gamma)}{\mathrm{sym}(\Gamma)} \; \lambda^{V(\Gamma)}$$

4) Extension to non-commutative coefficients

- Renormalization ruled by functors Diff and $\mathrm{Diff}_{\mathrm{CK}}$: same procedure for all QFTs!

- **Fermions** and **gauge bosons**: need **non commutative** algebra $A[\hbar]$ (at least $M_4(\mathbb{C})$), but the functor $\mathrm{Diff}: \mathcal{C}om \to \mathcal{G}roups$ does not apply!

- QED given by a commutative Hopf algebra [Van Suijlekom 2007] as matrix groups, but not functorial in A ($\bullet \neq$ convolution of Δ_{CK})!

- QED also given by non-commutative FdB Hopf algebra [Brouder-F-Krattenthaler 2006]:

$$H^{\mathrm{nc}}_{\mathrm{FdB}} = \mathbb{K}\langle x_n \mid n \geqslant 1 \rangle \qquad (x_0 = 1)$$
$$\Delta^{\mathrm{nc}}_{\mathrm{FdB}}(x_n) = \sum_{m+k_0+\cdots+k_m=n} x_m \otimes x_{k_0} \cdots x_{k_m}$$

- **Can we extend** Diff **to a functor on associative (non-commutative) algebras?**

  Not for free! If H and A are non-commutative, the convolution product

$$a * b = m_A \, (a \otimes b) \, \Delta_H \qquad \text{in} \qquad \mathrm{Hom}_{\mathcal{A}s}(H, A)$$

  is not well defined because $m_A : A \otimes A \to A$ **is not an algebra morphism**! (old problem)

Groups of series with coefficients in a non-commutative algebra A

- **Idea**: in $\mathcal{A}s$ replace the algebra $A \otimes B$ with product $(a \otimes b)\cdot(a' \otimes b') = aa' \otimes bb'$

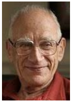
  by **free product algebra** $$A \amalg B = \mathbb{K} \oplus \bigoplus_{n\geqslant 1} \Big[\underbrace{A \otimes B \otimes A \otimes \cdots}_{n} \oplus \underbrace{B \otimes A \otimes B \otimes \cdots}_{n}\Big]$$

  with **concatenation** $$(a \otimes b)\cdot(a' \otimes b') = a \otimes b \otimes a' \otimes b'$$

  $\Longrightarrow$ $m_A : A \otimes A \to A$ lifts to **folding map** $\mu_A : A \amalg A \to A$ which is an algebra map!

- **Cogroup in** $\mathcal{A}s$ [Kan 1958, Eckmann-Hilton 1962] = associative algebra H with

comultiplication	$\Delta^{\amalg} : H \to H \amalg H$	coass.
counit	$\varepsilon : H \to \mathbb{K}$	+ prop
antipode	$S : H \to H$	+ prop



  $\Longrightarrow$ proalgebraic group $G(A) := \mathrm{Hom}_{\mathcal{A}s}(H, A)$ with $a * b = \mu_A\,(a \amalg b)\,\Delta^{\amalg}_H$

- **Group of invertible series:** $\mathrm{Inv}(A)$ $\Leftrightarrow$ $H = \mathbb{K}\langle x_1, x_2, \ldots\rangle$, $\Delta^{\amalg}(x_n) = \sum x_m \otimes x_{n-m}$

  [Brouder-F-Krattenthaler 2006]

  $\Longrightarrow$ good model for **renormalization factors** $Z(\lambda)$ in QFT!

5) When groups fail: use loops!

- **Problem**: if A is **not commutative**, the composition in $\mathrm{Diff}(A)$ is **not associative**:
$$\big(a \circ (b \circ c) - (a \circ b) \circ c\big)(\lambda) = (a_1 b_1 c_1 - a_1 c_1 b_1)\ \lambda^4 + O(\lambda^5) \neq 0$$

- **Loop** [Moufang 1935] = set Q with

multiplication	$a \cdot b$		(not nec. assoc.)
unit	1		+ cond.
left and right divisions	$a \backslash b$	a/b	+ cond.
$\Rightarrow$ left and right inverse of a	$1/a$	$a \backslash 1$	+ cond.

so that $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x = a \backslash b$, $y = b/a \in Q$

- **Associative loops** = groups $\quad 1/a = a\backslash 1 = a^{-1} \quad a\backslash b = a^{-1} \cdot b \quad a/b = a \cdot b^{-1}$

- **Smallest non-associative smooth loop:** $\mathbb{S}^7 = \{\text{unit octonions}\}$ (2-qbits!)

- **Thm.** [Sabinin 1977, 1981, 1986] Parallel transport along small geodesics gives a local smooth loop structure to any manifold M. Flat connection $\Rightarrow$ global loop.

- **Infinitesimal spaces:** Sabinin algebras (and Malt'sev algebras for Moufang loops). Differential calculus developed on smooth loops.

Loops of series with coefficients in a non-commutative algebra A

- **Coloop in $\mathcal{A}s$** [F-Shestakov 2019] = algebra H with

	comultiplication	$\Delta^{\amalg} : H \to H \amalg H$	(not nec. coass.)
	counit	$\varepsilon : H \to \mathbb{K}$	+ prop
	codivisions	$\delta_l, \delta_r : H \to H \amalg H$	+ prop
$\Rightarrow$	antipodes	$S_l, S_r : H \to H$	+ prop

$\Longrightarrow$ proalgebraic loop $Q(A) := \mathrm{Hom}_{\mathcal{A}s}(H, A)$ with $a * b = \mu_A \, (a \amalg b) \, \Delta^{\amalg}_H$

- **Loop of formal diffeomorphisms** [F-Shestakov 2019]:

$\mathrm{Diff}(A) \Leftrightarrow$

$$H = \mathbb{K}\langle x_1, x_2, \dots \rangle \qquad \Delta^{\amalg}(x_n) = \Delta^{\mathrm{nc}}_{\mathrm{FdB}}(x_n)$$

$\delta_r(x_n)$ = non-commutative Lagrange

$\delta_l(x_n)$ = new explicit formula (very complicated)

- **Thm.** In $\mathrm{Diff}(A)$ **inverse is unique** and $a/b(\lambda) = a \circ b^{-1}(\lambda)$ (while $a \backslash b(\lambda) \neq a^{-1} \circ b(\lambda)$!)

$\Rightarrow$ Dyson renormalization formulas make sense! cf. Birkhoff dec. $G = G^{ren} \bullet (\lambda_0, Z)^{-1}$

$\Rightarrow$ good model for **charge renormalization** $\lambda_0(\lambda)$ in QFT!

Free product is necessary!

In the loop $\mathrm{Diff}(A)$, we have $1/a = a\backslash 1 =: a^{-1}$ and also $a/b = a \circ b^{-1}$ but

$$a\backslash b \neq a^{-1} \circ b \; !$$

In the series $a\backslash b$, the coefficient

$$(a\backslash b)_3 = b_3 - \big(2a_1 b_2 + a_1 b_1^2\big) + \big(5a_1^2 b_1 + a_1 b_1 a_1 - 3a_2 b_1\big) - \big(5a_1^3 - 2a_1 a_2 - 3a_2 a_1 + a_3\big)$$

contains the term $\boxed{a_1 b_1 a_1}$ which can not be represented in the form

$$x(a) \otimes y(b) \in H^{\mathrm{nc}}_{\mathrm{FdB}} \otimes H^{\mathrm{nc}}_{\mathrm{FdB}},$$

while it can be represented as

$$x_1(a) \otimes y_1(b) \otimes x_1(a) \in H^{\amalg}_{\mathrm{FdB}} \amalg H^{\amalg}_{\mathrm{FdB}}.$$

This **justifies the need to replace** $\otimes$ **by** $\amalg$ in the coproduct and in the codivisions!

Conclusion:

- In pQFT, the renormalization group (RG) acts in a **functorial** way (via Hopf alg.): it gives the same procedure for any **scalar QFT**.
- The RG action can be **extended in a functorial way to non-scalar QFTs**, if we renounce to associativity in RG (by modifying the flow equations). Possible because diffeomorphisms form a non-associative loop with extra properties for which the RG action makes sense.

Perspectives:

- Proalgebraic groups and loops exist on associative, alternative, non-associative algebras (in particular unitary matrices): explore applications in maths and physics.
- Unitary loops on octonions are used to generalise gauge groups [Loginov 2003, Ootsuka-Tanaka-Loginov 2005]: explore the compatibility with non-associative RG.
- Develop software to compute with free product instead of tensor product.
- Compute a general integral formula for countertermes.
- Explore non-associative RG in Wilson's approach: replace usual flow of ODE by flow in smooth loops (cf. [Lev Sabinin 1999]).

Thank you for the attention!